Name: _____ RED ID: _____

Spring 2022 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please **print** your name and REDID in the designated spaces above. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. The first four questions are worth 5-11 points, and the remaining sixteen questions are worth 8-16 points. The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 8.5"x11" page (both sides) with your handwritten notes. This exam will begin at 10:30 and will end at 12:30; pace yourself accordingly. Good luck!

Special exam instructions for HH-130:

0. Exam seating: please do not sit next to anyone.
1. Please stow all bags/backpacks/boards at the front of the room. All contraband, except phones, must be stowed in your bag. All phones must be silent, non-vibrating, and either in your pocket or stowed in your bag.
2. Please remain quiet to ensure a good test environment for others.
3. Please keep your exam on your desk; do not lift it up for a better look.
4. If you have a question or need the restroom, please come to the front. Bring your exam.
5. If you are done and want to submit your exam and leave, please wait until one of the designated exit times, listed below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Designated exam exit times:
  10:50 "See you next semester"
  11:10 "I wish I had studied more"
  11:30 "One extra hour of drinking – worth it"
  11:50 "Maybe this will be good enough"
  12:10 "There is nothing more in my brain, let me out of here"
  12:30 "I need every second I can get"

Problem 1. Carefully state the following definitions/theorems:

   a. factorial

   b. tautology

Problem 2. Carefully state the following definitions/theorems:

   a. Proof by Contradiction Theorem (NOT the Contradiction semantic theorem)

   b. big O

Problem 3. Carefully state the following definitions/theorems:

   a. reflexive closure

   b. Chinese Remainder Theorem

Problem 4. Carefully state the following definitions/theorems:

   a. upper bound

   b. range

Problem 5. Let $x \in \mathbb{R}$ with $\lfloor x \rfloor = \lceil x \rceil$. Prove that $x \in \mathbb{Z}$.

Problem 6. State and prove the addition semantic theorem.

Problem 7. Prove or disprove the proposition: $\forall x \in \mathbb{N}, \ \exists y \in \mathbb{R}, \ x < y < 2x$.

Problem 8. Prove or disprove the proposition: $!n \in \mathbb{Z}, \ |3n - 3| = 6$.

Problem 9. Let $x \in \mathbb{R}$ with $x > -1$. Prove that: $\forall n \in \mathbb{N}_0, \ (1+x)^n \geq 1 + nx$.

Problem 10. Let $R = \{x \in \mathbb{Z} : 12|x\}$ and $S = \{x \in \mathbb{Z} : 4|x\}$. Prove or disprove that $R \subseteq S$.

Problem 11. Find all integers $x \in [0, 36)$ satisfying $15x \equiv 9 \pmod{36}$.

Problem 12. Consider the equivalence relations $\equiv_2$ and $\equiv_4$ on $\mathbb{Z}$. Prove or disprove the following relationship among equivalence classes: $[3]_{\equiv_4} \subseteq [1]_{\equiv_2}$.

Problem 13. Find a partial order on $S = \{a, b, c, d\}$ that has width 2 and height 2. Then find a linear extension of your partial order. Give both as Hasse diagrams, and clearly label which is which.

Problem 14. Define the relation R on $\mathbb{N}$ via $R = \{(x, y) : y = 7 - x\}$. (i) Determine if R is left total; (ii) Determine if R is right definite; (iii) Determine if R is a function. Be sure to justify your answers.

Problem 15. Set $S = \{x \in \mathbb{R} : x > 0\}$. Consider the function $f : \mathbb{R} \to S$ given by $f(x) = e^{-x/2}$. Prove that $f(x)$ is a bijection.

Problem 16. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\forall x \in \mathbb{R}, \ f(f(x)) = x$. Prove that f is a bijection.

For all the remaining problems, 17-20, recall that $\mathbb{Z}[x]$ is the set of polynomials, with integer coefficients, in variable x. For example, $3x^2 - x + 1 \in \mathbb{Z}[x]$ and $2 \in \mathbb{Z}[x]$.

Problem 17. Give an element of $2^{\mathbb{Z}} \times (\mathbb{Z}[x] \setminus \mathbb{Z})$.

Problem 18. Consider the relation R_1 on $\mathbb{Z}[x]$ given by $R_1 = \{(p(x), q(x)) : p(0) + q(0) = 0\}$. Prove or disprove that R_1 is an equivalence relation.

Problem 19. Consider the relation R_2 on $\mathbb{Z}[x]$ given below. Prove that R_2 is a partial order.
$R_2 = \{(p(x), q(x)) : \forall n \in \mathbb{Z}, \ p(n) \le q(n)\}$.

Problem 20. Consider the partial order R_2 on $\mathbb{Z}[x]$ from Problem 19. Prove or disprove that R_2 has a maximal element. If you forgot already, $R_2 = \{(p(x), q(x)) : \forall n \in \mathbb{Z}, \ p(n) \le q(n)\}$.